A Logic of Expertise

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- **This work:** a modal logic framework to reason about expertise
- **Key questions:**
  - What does it mean for an information source to be an expert on a formula φ?
  - What information can we infer from non-expert reports?
  - How does expertise relate to knowledge?
- **Main contributions and results:**
  - Syntax and semantics
  - Link between logic of expertise and S5 epistemic logic
  - Sound and complete axiomatisation

Outline

Motivating Example

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- Consider an economist reporting on COVID-19 vaccine rollout

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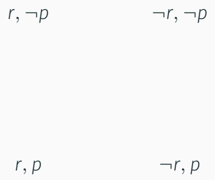

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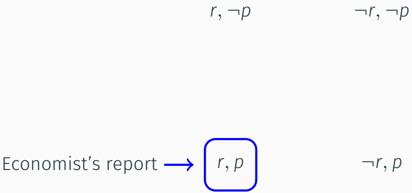

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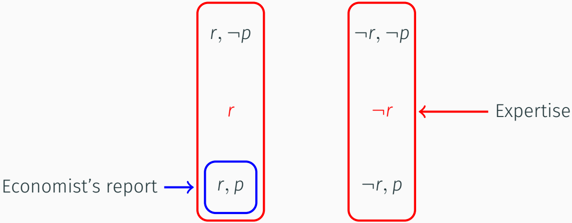

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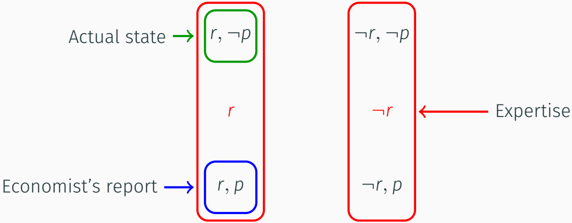

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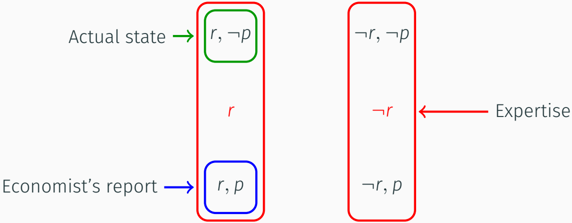


- Economist's report is **false**, but 'true' after accounting for lack of expertise on p: it is sound

Syntax and Semantics

Syntax

- Fix one unnamed information source
- Language $\mathcal{L}$ formed by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{E}\varphi \mid \mathsf{S}\varphi \mid \mathsf{U}\varphi$$

- $\mathsf{E}\varphi$: the source has expertise on φ
- $\mathsf{S}\varphi$: φ is sound for the source to report, i.e. it is true up to lack of expertise
- $\mathsf{U}\varphi$: φ holds in all possible states

Example

In the economist example we had $\mathsf{E}r$, $\neg\mathsf{E}p$ and $\mathsf{S}(r \wedge p)$

Semantics

- Expertise formulas are interpreted using a special case of neighbourhood semantics
- A model is a triple $M = (X, P, v)$, where
  - X is a set of states
  - $v : \text{Prop} \to 2^X$ is a valuation function: $v(r) \subseteq X$
  - $P \subseteq 2^X$ is an expertise set
- **Intuition:** $A \in P$ iff source has expertise to tell whether or not the actual state is in A
- Basic constraints on P:

(P1) $X \in P$ (source has expertise on tautlogies)
(P2) If $A \in P$ then $X \setminus A \in P$ (expertise is symmetric w.r.t. negation)
(P3) If $\{A_i\}_{i \in I} \subseteq P$, then $\bigcap_{i \in I} A_i \in P$ (expertise is closed under conjunctions)

Semantics (cont'd.)

- Truth conditions:

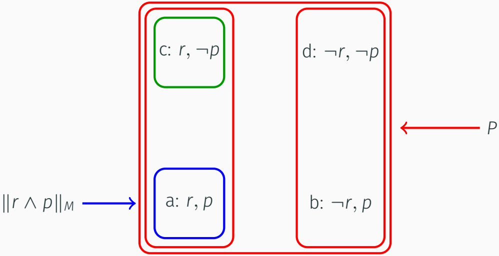
$$
\begin{array}{lcl}
M, x \vDash p & \iff & x \in v(p) \\
M, x \vDash \neg\varphi & \iff & M, x \nvDash \varphi \\
M, x \vDash \varphi \wedge \psi & \iff & M, x \vDash \varphi \text{ and } M, x \vDash \psi \\
M, x \vDash \mathrm{E}\varphi & \iff & \|\varphi\|_M \in P \\
M, x \vDash \mathrm{S}\varphi & \iff & \text{for all } A \in P,\ \|\varphi\|_M \subseteq A \text{ implies } x \in A \\
M, x \vDash \mathrm{U}\varphi & \iff & \text{for all } y \in X,\ M, y \vDash \varphi
\end{array}
$$

where $\|\varphi\|_M = \{x \in X \mid M, x \vDash \varphi\}$

- **Note:**
  - $\mathrm{S}\varphi$ is true iff x is contained in all supersets of $\|\varphi\|_M$ on which the source has expertise
  - Truth value of $\mathrm{E}\varphi$ does not depend on the "actual" state x

Example revisited

- $X = \{a, b, c, d\}$
- $v(r) = \{a, c\}$ and $v(p) = \{a, b\}$
- $P = \{\emptyset, \{a, c\}, \{b, d\}, X\}$.



- We have

$$\begin{array}{ll} M \models \mathrm{E}r & M, c \models \neg(r \wedge p) \\ M \models \mathrm{E}\neg r & M, c \models \mathrm{S}(r \wedge p) \\ M \models \neg Ep & M, c \models r \end{array}$$

Some important validities and non-validities

- $\mathsf{E}\varphi \equiv \mathsf{E}\neg\varphi$
- Either $M \models \mathsf{E}\varphi$ or $M \models \neg\mathsf{E}\varphi$
- $\models (\mathsf{E}\alpha \wedge \mathsf{E}\beta) \rightarrow \mathsf{E}(\alpha \wedge \beta)$
- $\models \varphi \rightarrow \mathsf{S}\varphi$
- If $\models \alpha \rightarrow \beta$ then $\models (\mathsf{S}\alpha \wedge \mathsf{E}\beta) \rightarrow \beta$
    - If α is sound, any logically weaker formula β on which the source has expertise must be true
- E is non-normal: $\mathsf{E}(\alpha \rightarrow \beta) \rightarrow (\mathsf{E}\alpha \rightarrow \mathsf{E}\beta)$ is not in general valid

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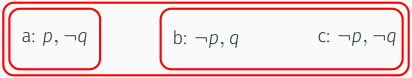

$M \vDash \mathsf{E}p \wedge \mathsf{E}(p \rightarrow q) \wedge \neg\mathsf{E}q$

Connection with S5

Expertise and knowledge

- What is the relation between expertise and knowledge?
- S5 epistemic logic: knowledge operator K; axiomatised by the KT5 axioms:

(K)	$\mathrm{K}(\varphi \rightarrow \psi) \rightarrow (\mathrm{K}\varphi \rightarrow \mathrm{K}\psi)$	(distribution axiom)
(T)	$\mathrm{K}\varphi \rightarrow \varphi$	(knowledge is true)
(5)	$\neg\mathrm{K}\varphi \rightarrow \mathrm{K}\neg\mathrm{K}\varphi$	(negative introspection)

- S5 is the logic of Kripke frames whose accessibility relation is an equivalence relation
- **S5 model:** $M' = (X, R, v)$, where $R \subseteq X \times X$ is an equivalence relation

Expertise and knowledge (cont'd.)

- Translation t from $\mathcal{L}$ into $\mathcal{L}_{\mathsf{KU}}$:

$$
\begin{array}{lll}
t(p) = p; & t(\neg\varphi) = \neg t(\varphi); & t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi); \\
t(\mathsf{U}\varphi) = \mathsf{U}t(\varphi); & t(\mathsf{E}\varphi) = \mathsf{U}(t(\varphi) \to \mathsf{K}t(\varphi)); & t(\mathsf{S}\varphi) = \neg\mathsf{K}\neg t(\varphi)
\end{array}
$$

Theorem

Any expertise model $M = (X, P, v)$ uniquely determines an S5 model M^* such that for any $\varphi \in \mathcal{L}$,

$$
\underbrace{M, x \vDash \varphi}_{\textit{expertise semantics}} \iff \underbrace{M^*, x \vDash t(\varphi)}_{\textit{Kripke semantics}}
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- (the converse also holds)

Expertise and knowledge (cont'd.)

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- For propositional φ:
  - $\mathsf{E}\varphi \overset{t}{\longmapsto} \mathsf{U}(\varphi \to \mathsf{K}\varphi)$:
    in all possible states, if φ were true the source would know it
  - $\mathsf{S}\varphi \overset{t}{\longmapsto} \neg\mathsf{K}\neg\varphi$:
    the source does not *know* that φ is false
- The equivalence allows E to be expressed in terms of S and U:

$$\mathsf{E}\varphi \equiv \mathsf{U}(\mathsf{S}\varphi \to \varphi)$$

$$\neg\mathsf{E}\varphi \equiv \hat{\mathsf{U}}(\mathsf{S}\varphi \wedge \neg\varphi) \equiv \hat{\mathsf{U}}(\varphi \wedge \mathsf{S}\neg\varphi)$$

Axiomatisation

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- We showed a close link between the logic of expertise and S5
- S5 is axiomatised by KT5
- Can we use this to obtain an axiomatisation of the logic of expertise frames?

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- Can we use this to obtain an axiomatisation of the logic of expertise frames?
- **Yes:** let L be the extension of propositional logic with the following axioms and inference rules:

(K_S)	$(S\varphi \wedge \neg S\psi) \rightarrow S(\varphi \wedge \neg\psi)$	(K_U)	$U(\varphi \rightarrow \psi) \rightarrow (U\varphi \rightarrow U\psi)$
(T_S)	$\varphi \rightarrow S\varphi$	(T_U)	$U\varphi \rightarrow \varphi$
(5_S)	$S\neg S\varphi \rightarrow \neg S\varphi$	(5_U)	$\neg U\varphi \rightarrow U\neg U\varphi$
(ES)	$E\varphi \leftrightarrow U(S\varphi \rightarrow \varphi)$	(Inc)	$U\varphi \rightarrow \neg S\neg\varphi$

(MP)	From φ and $\varphi \rightarrow \psi$ infer ψ
(Nec_U)	From φ infer $U\varphi$
(R_S)	From $\varphi \leftrightarrow \psi$ infer $S\varphi \leftrightarrow S\psi$

Axiomatisation (cont'd.)

Theorem

L *is sound and complete w.r.t. the class of expertise frames.*

Soundness is mostly routine. Completeness is shown in 3 steps:

1. L − (ES) is complete w.r.t. augmented expertise frames[1,2], for the fragment $\mathcal{L}_{\mathsf{SU}}$
   - Use an equivalence relation R_{U} to interpret the universal modality
   - Use standard canonical model construction + earlier results
2. L − (ES) is complete w.r.t. expertise frames, for $\mathcal{L}_{\mathsf{SU}}$
   - Follows from step (1) by taking generated sub-frames
3. L is complete w.r.t. expertise frames for the full language $\mathcal{L}$
   - Follows from (2) since (ES) allows $\mathcal{L}$ to be reduced to the fragment $\mathcal{L}_{\mathsf{SU}}$

[1] Valentin Goranko and Solomon Passy. "Using the Universal Modality: Gains and Questions". In: *Journal of Logic and Computation* (1992).

[2] Giacomo Bonanno. "A simple modal logic for belief revision". In: *Synthese* (2005).

Conclusion

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- Summary:
  - Introduced a modal logic framework for expertise and soundness
  - Established a connection with S5 epistemic logic, which led to a sound and complete axiomatisation
- Future work:
  - Estimate source's expertise
  - Probabilistic or graded interpretation of expertise
  - What is the relation between expertise and trust?
  - With evidence?
  - How does expertise change over time?